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DISCRETE MATHEMATICS
Discrete Mathematics 306 (2006) 666–669
www.elsevier.com/locate/disc

Note

Linkability in iterated line graphs

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Received 25 February 2004; received in revised form 2 November 2005; accepted 17 November 2005

Abstract

We prove that for every graph H with the minimum degree $\delta \geqslant 5$, the third iterated line graph $L^3(H)$ of H contains $K_{\delta\lfloor\sqrt{\delta-1}\rfloor}$ as a minor. Using this fact we prove that if G is a connected graph distinct from a path, then there is a number k_G such that for every $i \geqslant k_G$ the i-iterated line graph of G is $\frac{1}{2}\delta(L^i(G))$-linked. Since the degree of $L^i(G)$ is even, the result is best possible.

Keywords: Iterated line graph; k-linked graph; Connectivity; Graph dynamics; Minor

1. Introduction and results

Let G be a graph. Its *line graph* $L(G)$ is defined as the graph whose vertices are the edges of G, with two vertices adjacent if and only if the corresponding edges are adjacent in G. Although the line graph operator is one of the most natural ones, only in recent years there is recorded a larger interest in studying iterated line graphs. *Iterated line graphs* are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

The diameter and radius of iterated line graphs are examined in [10], while [7] is devoted to the centers of these graphs. In [3] and [2], Hartke and Higgins study the growth of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [6], while in [13] , Xiong and Liu characterize the graphs whose i-iterated line graphs are Hamiltonian.

Note that the i-iterated line graph of a path on n vertices is a path on $n-i$ vertices for $i<n$ and an empty graph if $i \geqslant n$. The iterated line graph of a cycle is isomorphic to the original cycle, and each iterated line graph of a claw $K_{1,3}$ is isomorphic to a triangle. Hence, it suffices to study connected graphs distinct from paths, cycles and the claw $K_{1,3}$. Such graphs are called *prolific*, since every two members of the sequence $\{L^i(G)\}_{i=0}^{\infty}$ are non-isomorphic.

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[1] Supported by Kuwait University Grant #SM 02/00.

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doi:10.1016/j.disc.2005.11.018

Let $\delta(H)$ denote the minimum degree of H. In [3], we have:

Theorem A. *Let G be a prolific graph. Then there is i_G such that for every i, $i \geqslant i_G$, holds that*

$$\delta(L^{i+1}(G)) = 2 \cdot \delta(L^i(G)) - 2.$$

Obviously, $\delta(L^{i_G}(G)) \geqslant 3$ in the above theorem. As a consequence, by the results of [6], we obtain:

Proposition B. *Let G be a prolific graph. Then for every i, $i \geqslant i_G + 5$, the connectivity of $L^i(G)$ equals the minimum degree of $L^i(G)$.*

Here, i_G is the constant appearing in Theorem A.

In this paper, we study the linkability of iterated line graphs. A graph with at least $2k$ vertices is said to be *k-linked* if for every $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ it contains k vertex-disjoint paths $P_1, P_2, \ldots, P_k$, such that P_i connects s_i to t_i, $1 \leqslant i \leqslant k$.

Obviously, if a graph is k-linked, then it is k-connected. The converse is far from being true. Jung [4] and, independently, Larman and Mani [8] proved that every $2k$-connected graph that contains a subgraph isomorphic to a subdivision of K_{3k} is k-linked. This together with a result of Mader [9] implies that for every k there is an $f(k)$ such that every $f(k)$-connected graph is k-linked. Robertson and Seymour [11] extended the result of Jung et al. As a consequence of Theorem (5.4) of [11] we have:

Proposition C. *Every $2k$-connected graph that has a K_{3k}-minor is k-linked.*

In [1], Bollobás and Thomason proved that every $2k$-connected graph G with at least $11k|V(G)|$ edges is k-linked. This implies that every $22k$-connected graph is k-linked. Recently, Thomas and Wollan [12] improved the lower bound on the number of edges in the Bollobás and Thomason result to $8k|V(G)|$. This was further improved by Kawarabayashi et al. [5]. They showed that every $2k$-connected graph with average degree at least $12k$ is k-linked. Consequently, every $12k$-connected graph is k-linked.

Our main result is the following theorem:

Theorem 1. *Let G be a prolific graph. Then there is k_G such that for every $i \geqslant k_G$ the graph $L^i(G)$ is $\frac{1}{2}\delta(L^i(G))$-linked.*

Observe that a graph with minimum degree δ cannot be more than $\frac{1}{2}\delta$-linked if δ is even. (Consider $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ where s_k is a vertex of minimum degree $\delta = 2k - 2$, and $s_1, \ldots, s_{k-1}, t_1, \ldots, t_{k-1}$ are all of the neighbors of s_k.) Since the minimum degree of iterated line graph $L^i(G)$ is even if i is "big enough", the result of Theorem 1 is best possible.

We mention that it is an open problem to find "good" bounds in terms of G on the numbers i_G and k_G in Theorems A and 1, respectively. However, if the graph G is regular of degree δ, then from the proof of Theorem 1 it can be deduced that $k_G \leqslant 11$.

In the proof of Theorem 1, which is trivially true for cycles and the claw $K_{1,3}$, we use the following statement:

Theorem 2. *Let H be a graph with a minimum degree $\delta \geqslant 5$. Then $L^3(H)$ has K_t as a minor, $t = \delta \cdot \lfloor\sqrt{\delta - 1}\rfloor$.*

We remark that the best lower bound for the size of a complete graph in $L^3(H)$ is $4\delta - 6$. Theorem 2 shows that there exists a much larger complete graph as a minor.

2. Proofs

Let G be a graph and let v be a vertex of $L^k(G)$, $k \geqslant 1$. Then v corresponds to an edge of $L^{k-1}(G)$, and this edge will be called 1-history of v. For $i \geqslant 2$ we define i-histories recursively. The i-history of v is a subgraph of $L^{k-i}(G)$, edges of which are induced by the vertices of $L^{k-i+1}(G)$ which are in $(i-1)$-history of v.

Observe that 1-history is always an edge and 2-history is a path of length two. The situation is more complicated for i-histories when $i \geqslant 3$. The only fact we can say is that i-history is a connected graph with at most i edges, distinct from any path with less than i edges, see [10]. Therefore, we do not visualize the vertices of $L^3(H)$ in H using their 3-histories in the proof of Theorem 2. First, we use 2-histories of vertices of $L^2(H)$ and subsequently 1-histories of vertices of $L^3(H)$. In such a way, vertices of $L^3(H)$ correspond to pairs of "adjacent" 2-histories in H.

We prove Theorem 2 in a slightly stronger form. We prove that for an arbitrary vertex v of H there is a subgraph K of $L^3(H)$, such that K_t is a minor of K and the 3-history of every vertex of K contains v.

Proof of Theorem 2. Denote by $v_1, v_2, \ldots, v_\delta, \ldots$ the neighbors of v in H.

Consider 2-histories of the vertices of $L^2(H)$ in H. Denote by $c_{i,i'}$ the vertex of $L^2(H)$ with 2-history $(v_i, v, v_{i'})$, and denote by C the set of these vertices. Then $|C| \geqslant \binom{\delta}{2}$. Denote by A_i those vertices of $L^2(H)$, whose 2-history have v_i as a central vertex and v as an endvertex. Observe that $|A_i| \geqslant \delta - 1$, the vertices of A_i induce a complete graph in $L^2(H)$, and they are adjacent to all $c_{i,i'}$, $i' \neq i$. Moreover, the sets $A_1, A_2, \ldots, A_\delta$ are mutually disjoint.

Let $s = \lfloor \sqrt{\delta - 1} \rfloor$. Equitably partition every A_i into s parts $A_{i,1}, A_{i,2}, \ldots, A_{i,s}$, so that $-1 \leqslant |A_{i,j}| - |A_{i,j'}| \leqslant 1$ for every $j \neq j'$. Then each $A_{i,j}$ contains at least s vertices, and as $\delta \geqslant 5$, we have $s \geqslant 2$. Denote the vertices of $A_{i,j}$ by $a_{i,j,1}, a_{i,j,2}, \ldots, a_{i,j,s}, \ldots$.

Now denote by $X_{i,j}$ the set of those vertices of $L^3(H)$, whose 1-histories in $L^2(H)$ contain only the vertices of $A_{i,j}$. In the following we show that there are internally vertex-disjoint paths in $L^3(H)$ connecting the sets $X_{i,j}$. Let $X_{i,j}$ and $X_{i',j'}$ be two such sets, $(i, j) \neq (i', j')$. There are two cases to distinguish:

Case 1: $i = i'$. We join the vertex of $X_{i,j}$ with 1-history $(a_{i,j,1}, a_{i,j,2})$ with the vertex of $X_{i,j'}$ with 1-history $(a_{i,j',1}, a_{i,j',2})$ by a path of length two. Its interior vertex has 1-history $(a_{i,j,1}, a_{i,j',1})$.

Case 2: $i \neq i'$. We join the vertex of $X_{i,j}$ with 1-history $(a_{i,j,1}, a_{i,j,j'})$ with a vertex of $X_{i',j'}$ with 1-history $(a_{i',j',1}, a_{i',j',j})$ by a path of length three. Its interior vertices have 1-histories $(a_{i,j,j'}, c_{i,i'})$ and $(c_{i,i'}, a_{i',j',j})$.

Obviously, the paths just constructed in $L^3(H)$ are disjoint. If we contract the vertices of $X_{i,j}$ into a single vertex $x_{i,j}$, $1 \leqslant i \leqslant \delta$ and $1 \leqslant j \leqslant s$, then the vertices $x_{i,j}$ together with the constructed paths form a subdivision of $K_{\delta \cdot s}$. Now the result is a consequence of the fact that all the vertices in $X_{i,j}$ and in the paths have v in their 3-history. □

We remark that if $|A_{i,j}| = s$ in the previous proof, then the paths from $X_{i,j}$ to $X_{i',j'}$, where $i \neq i'$ and $j' = 1, 2, \ldots, s$, exhaust all the vertices with 1-histories $(a_{i,j,\cdot}, c_{i,i'})$. This means that the choice $s = \lfloor \sqrt{\delta - 1} \rfloor$ is optimal if we restrict ourselves to the types of paths described in Cases 1 and 2.

Notice that the proof of Theorem 2 implies that, if T is a tree with a central vertex v, such that v and its neighbors have degree δ and all the remaining vertices are pendant, then $L^3(T)$ has K_t as a minor, $t = \delta \cdot \lfloor \sqrt{\delta - 1} \rfloor$.

Proof of Theorem 1. Choose k_G such that

$$k_G \geqslant i_G + 5$$

and

$$\lfloor \sqrt{\delta(L^{k_G - 3}(G)) - 1} \rfloor \geqslant 12,$$

where i_G is the constant from Theorem A. Then for every $i \geqslant k_G$, it follows from Proposition B that $L^i(G)$ is $\delta(L^i(G))$-connected. Further, by Theorem A we have $\delta(L^i(G)) = 8\delta(L^{i-3}(G)) - 14$. Finally, by Theorem 2 $L^i(G)$ has a K_t-minor with

$$t = \delta(L^{i-3}(G)) \lfloor \sqrt{\delta(L^{i-3}(G)) - 1} \rfloor \geqslant \tfrac{1}{8}(\delta(L^i(G)) + 14) \cdot 12 > \tfrac{3}{2}\delta(L^i(G)).$$

By Proposition C this implies that $L^i(G)$ is $\delta(L^i(G))/2$-linked. □

Acknowledgment

Part of this work was done during a visit by the second author to the Technische Universität in Ilmenau. This stay was sponsored by the Slovak-German Joint Research Fund managed by the Ministry of Education of Slovak Republic.

References

[1] B. Bollobás, A. Thomason, Highly linked graphs, Combinatorica 16 (1996) 313–320.
[2] S.G. Hartke, A.W. Higgins, Maximum degree growth of the iterated line graph, Electron. J. Combin. 6 (1) (1999) 9.
[3] S.G. Hartke, A.W. Higgins, Minimum degree growth of the iterated line graph, Ars Combin. 69 (2003) 275–283.
[4] H.A. Jung, Verallgemeinerung des n-Fachen Zusammenhangs fuer Graphen, Math. Annalen 187 (1970) 95–103.
[5] K.-I. Kawarabayashi, A. Kostochka, G. Yu, On sufficient degree conditions for a graph to be k-linked, Combin. Probab. Comput., to appear.
[6] M. Knor, L'. Niepel, Connectivity of iterated line graphs, Discrete Appl. Math. 125 (2003) 255–266.
[7] M. Knor, L'. Niepel, L'. Šoltés, Centers in iterated line graphs, Acta Math. Univ. Comenian. LXI 2 (1992) 237–241.
[8] D.G. Larman, P. Mani, On the existence of certain configurations within graphs and the 1-skeletons of polytopes, Proc. London Math. Soc. 20 (1974) 144–160.
[9] W. Mader, Homomorphieeigenschaften und Mittlere Kantendichte von Graphen, Math. Annalen 174 (1967) 265–268.
[10] L'. Niepel, M. Knor, L'. Šoltés, Distances in iterated line graphs, Ars Combin. 43 (1996) 193–202.
[11] N. Robertson, P.D. Seymour, Graph Minors XIII. The disjoint paths problem, J. Combin. Theory (B) 63 (1995) 65–110.
[12] R. Thomas, P. Wollan, An improved extremal function for graph linkages, European J. Combin. 26 (2005) 309–324.
[13] L. Xiong, Z. Liu, Hamiltonian iterated line graphs, Discrete Math. 256 (2002) 407–422.